By some reaction of this type, if proper protection against intramolecular rearrangement be provided, we may predict that a di-aryl substituted hydrogen peroxide will be prepared, which in turn will dissociate into mono-aryl oxide, ArO, the odd molecule, or free radical, of univalent oxygen. Such a peroxide would probably be less dissociated than the similarly substituted hydrazine, just as the latter is less dissociated than the corresponding ethane.

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NEWTON'S METHOD IN GENERAL ANALYSIS

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The present paper is essentially an extension of the methods and results given by Dean H. B. Fine, On Newton's Method of Approximation,[1] and the results there obtained need not be explained here.

An illustration of some general notions.—As might be expected, Newton's Method is of very wide application and may be used, for example in the following three cases to find a real 'root' of the real functionelle $F[x(s)]$ where F has the property that in a certain domain, and for every $x(s)$ and $x(s) + h(s)$ in this domain, there exist functionelles F_1 and F_2 for which

$$\begin{aligned} F[x(s)+h(s)] \equiv F[x(s)] &+ \int_0^1 F_1[x(s), r]h(r)dr \\ &+ \int_0^1 \int_0^1 \tfrac{1}{2} F_2[\xi(s), r_1, r_2]\, h(r_1)\, h(r_2)\, dr_1 dr_2, \end{aligned} \tag{A}$$

(1) where $\xi(s)$ is such that $\max_s | \xi(s)-x(s) | \leqq \max_s | h(s) |$ and $\max_s | x(s)+h(s)-\xi(s) | \leqq \max_s | h(s) |$, if by '$\max_s$' is meant the maximum as s varies, and by a 'root' of the above equation is meant a function $x(s)$, such that $\max_s | F[x(s)] | = 0$; or (2) where $\xi(s)$ is such that $\sqrt{\int_0^1[\xi(r)-x(r)]^2 dr} \leqq \sqrt{\int_0^1 h^2(r)\, dr}$ and $\sqrt{\int_0^1[x(r)+h(r)-\xi(r)]^2 dr} \leqq \sqrt{\int_0^1 h^2(r)\, dr}$, and by a 'root' is meant a function $x(s)$, such that $\sqrt{\int_0^1 F[x(r)]^2 dr} = 0$; or (3), where $\xi(s)$ is such that $\int_0^1 | \xi(r)-x(r) | dr \leqq$

$\int_0^1 | h(r) | dr$, and $\int_0^1 | x(r) + h(r) - \xi(r) | dr \leqq \int_0^1 | h(r) | dr$ and by a 'root' is meant a function $x(s)$ such that $\int_0^1 | F[x(r)] dr = 0$.

The above equation (A) we can write symbolically in the form

$$F(x + h) = F(x) + F'(x)h + \tfrac{1}{2}F''(\xi)h^2. \qquad (A')$$

We must regard x, h, and ξ as functions whose independent variables range over the same interval, viz., from zero to one. Any linear transformation of x must be accompanied by the same transformation on h and ξ. We shall speak of x, h, and ξ as covariant and of equal weight, viz., *one*, on the first of the two ranges that we shall consider. On the other hand the symbol h^2 or $h(r_1)\ h(r_2)$ is covariant with h but of weight *two*. Now h and F are not covariant, but *divariant*. We shall say that F is defined on the second range. Clearly F, F' and $\frac{1}{2} F''$ are covariant and of weight one on this second range. Similarly F' and h are *contravariant*, for only if F' be subjected to the transformation contragredient to that on h, will the term $F'h$ be left invariant. Similarly $\frac{1}{2} F''$ must be subjected to the square of the contragredient transformation. The whole situation may be succinctly expressed by introducing the notion of *signature*, which denotes the range and weight and whether covariant or contravariant. We shall say that x, h, and ξ are of signature (1, 0) being defined on the first but not on the second range, and of weight one on the first, h^2 is of signature (2,0) being of weight two. F is of signature (0, 1), F' of signature $(-1, 1)$ being contravariant with functions of weight one on the first range, and covariant with functions of weight one on the second, $\frac{1}{2} F''$ is of signature $(-2, 1)$. A constant we may speak of as of signature (0, 0).

The three expressions (1) $\max_s | z(s) |$, (2) $\sqrt{\int_0^1 z^2(r)dr}$, (3) $\int_0^1 | z(r) | dr$ we may subsume under the notation $\| z \|$ for the ranges (1, 0) and (0, 1). If the integration be in the sense of Lebesgue, it is true that if $z(s) = 0$ identically in s, then $\| z \| = 0$ in each of the cases, but in the second and third cases it does not follow from $\| z \| = 0$ that $z(s) = 0$ identically in s.

We may speak of x, h, ξ, F, F', $\frac{1}{2} F''$ as vectors, this being an extension of the familiar notion of a real vector in three dimensions. The symbolic expression (A') brings in symbolic products of apparently different types. But if the signature be kept in mind, no ambiguity results, for in each case, the product of two vectors has for its signature, the matrical sum of the signatures of the factors. For example in $\frac{1}{2} F''(\xi)h^2$, we have $(-2, 1)+(1, 0)+(1, 0) = (0, 1)$ as desired. The norm $\| z \|$ of the vector z may also be readily defined for h^2 F' and $\frac{1}{2} F''$, so

as to satisfy certain useful inequalities. We shall define in the three cases respectively; (1) $\| h^2 \| \equiv \max_{s_1 s_2} | h(s_1)\, h(s_2) |$, $\| F'[x(s), r] \| \equiv \max_s \int_0^1 | F[x(s), r] |\, dr$, $\| \tfrac{1}{2} F''[x(s), r_1, r_2] \| \equiv \max_s \int_0^1 \int_0^1 | \tfrac{1}{2} F''[x(s), r_1, r_2] |\, dr_1\, dr_2$; (2) $\| h^2 \| \equiv \int_0^1 \int_0^1 h^2(r_1)\, h^2(r_2)\, dr_1\, dr_2$, $\| F'[x(s), r] \| \equiv \sqrt{\int_0^1 \int_0^1 F'[x(s), r]^2\, ds\, dr}$, $\| \tfrac{1}{2} F''[x(s), r_1, r_2] \| \equiv \sqrt{\int_0^1 \int_0^1 \int_0^1 \{\tfrac{1}{2} F''[x(s), r_1, r_2]\}^2\, ds\, dr_1\, dr_2}$; (3) $\| h^2 \| \equiv \int_0^1 \int_0^1 | h(r_1)\, h(r_2) |\, dr_1\, dr_2$, $\| F'[x(s), r] \| \equiv \int_0^1 \{\max_r | F[x(s), r] | \}\, ds$, $| \tfrac{1}{2} F''[x(s), r_1, r_2] \| \equiv \int_0^1 \{\max_{r_1 r_2} | \tfrac{1}{2} F''[x(s), r_1, r_2] | \}\, ds$. With these definitions $\| z_1 + z_2 \| \leqq \| z_1 \| + \| z_2 \|$ and $\| z_1 z_2 \| \leqq \| z_1 \|\, \| z_2 \|$, where $z_1 z_2$ means the symbolic product. Furthermore we may define $\| z \|$ for z of range (0, 0) as identical with $| z |$.

The case of Newton's method for one variable and n variables has been discussed by Dean Fine in the paper referred to. These cases and the case of integration in which the ranges are continuous may be treated by the methods discussed in the present paper. But the conditions here used are of an abstract sort and may be used in much more extensive cases as will be clear to those familiar with the recent work of E. H. Moore, M. Fréchet, F. Riesz, V. Volterra, etc. The scalars, or vectors of signature (0, 0), which in all of the classical instances are ordinary real numbers, may be taken as Hensel p-adic numbers or elements in any *perfekte bewertete Körper* of Kürschák,[2] but more generally do not need to constitute a field as division is not essential. To avoid repetition the further discussion of this case will not be treated separately but may be regarded as included in the following sections.

Preliminary concepts.—Starting with an arbitrarily chosen range, which we shall refer to by the signature (1, 0), we shall suppose that we may construct ranges of the signatures (0, 0), (0, 1), (1, 1), (−1, 1), (−2, 1), (2, 0), respectively, where the range of signature (0, 0) contains but one element. An explicit definition of signature will not be required.

By the term *vector* or *function on a range*, will be meant a correspondence from the elements of a range to a set of *scalars* where each element determines one and only one scalar. A vector of signature (0, 0) is by definition a scalar, and conversely. The sum and the difference of two scalars will be required to exist uniquely, addition being associative. The sum and the difference of two vectors will be the usual vector or matrix sum and difference, respectively, and will be required to exist uniquely in the cases considered. The product of two vectors will be required to exist uniquely in the cases considered, and to be (1) associative, (2) completely distributive with respect to addition—so that addition is proved to be commutative, (3) continuous in each factor, continuity being defined as below, and (4) such that the signature of

the range of the product is the matrix sum of the signatures of the ranges of the factors.

The notation $\| z \|$ will be used for the *norm* of z. We shall require only that (1) $\| z \|$, $= \| -z \|$, is a uniquely defined real non-negative number such that $z \equiv 0$ implies $\| z \| = 0$, but not necessarily conversely; (2) $\| z_1 + z_2 \| \leqq \| z_1 \| + \| z_2 \|$, $\| z_1 z_2 \| \leqq \| z_1 \| \ \| z_2 \|$, and in particular, if $\| z_1 \| = 0$, $\| z_1 + z_2 \| = \| z_2 \|$, the sum and the product being defined as above; (3) the signature of z being given and a positive number ϵ, being given, there exists a z, such that $0 < \| z \| < \epsilon$; and (4) if $z_1, z_2, \ldots, z_n, \ldots$ be a denumerable sequence of vectors on the same range, and such that $\| z_1 \| + \| z_2 \| + \ldots + \| z_n \| + \ldots$ converges to a finite limit, then there is at least one z on the same range such that given any positive number ϵ, there exists a number m such that $n > m$ implies that $\| z - \Sigma_1^n z_i \| < \epsilon$ and $\lim_{n=\infty} \| \Sigma_1^n z_i \| = \| (z) \|$. Such a z will be called a *limit* of $z_1 + z_2 + \ldots + z_n + \ldots$, and we notice that for any two such limits z and z', $\| z - z' \| = 0$. We shall define two vectors z and z' for which $\| z - z' \| = 0$ as equivalent. It is by virtue of this last-named property that the vectors of a given range form in some sense a closed set.

A vector b with the signature $(-1, 1)$ will be said to be *non-singular*, if and only if there exists a unique vector b^{-1}, with the signature $(1, -1)$ such that the equation $a = bx$ where a is of signature $(0, 1)$, and x is required to be of signature $(1, 0)$, always has one and only one solution given by $x = b^{-1}a$.

We shall extend the usual definition of continuity to the case where $g(x)$ and x are of any signature as follows:—The function $g(x)$ is *continuous* in x at x' if for every assigned positive constant ϵ, there exists a positive constant δ, such that $\| x - x' \| < \delta$ implies that $\| g(x) - g(x') \| < \epsilon$.

Description of Newton's method.—In describing Newton's Method, we presuppose that we are given initially a function $f(x)$, where x is of signature $(1, 0)$ and $f(x)$ of signature $(0, 1)$, and such that we may expand $f(x+h)$ as follows:—

$$f(x+h) = f(x) + f'(x)h + \tfrac{1}{2} f''(\xi)h^2, \tag{1}$$

where h and ξ are of signature $(1, 0)$, h^2 of signature $(2, 0)$, f' of signature $(-1, 1)$, f'' of signature $(-2, 1)$, the expansion being valid in a known domain. The ξ is supposed to be dependent on the x and h, but such that simultaneously $\| \xi - x \| \leqq \| h \|$ and $\| (x+h) - \xi \| \leqq \| h \|$. The functions f, f', and f'' are all supposed to be continuous in the domain considered. Equation (1) may also be written, for convenience, in the form $f(x+h) = a + bh + ch^2$.

Newton's Method consists in the following steps. Choose any x_0 in the domain and then select h_0 as a solution of $a_0+b_0h_0=0$. Put $x_1=x_0+h_0$, and repeat. Thus $a_i+b_ih_i=0$, and $x_{i+1}=x_i+h_i$, $i=0, 1, \ldots$, where in general a_i, b_i, c_i mean $a(x_i)$, $b(x_i)$, $c(x_i)$ respectively. If $\lim_{i=\infty} x_i$ exists uniquely, $(=x')$, and is in the domain, then x' is a root of $f(x)=0$.

In order that Newton's Method may be applied, we note that the equations $a_i+b_ih_i=0$ must be solved uniquely for h_i, so that $b(x)$ must be non-singular in the domain. We shall write $\| a_i\| =\alpha_i$, $\| b_i^{-1} \| = 1/\beta_i$, $\| c_i \| = \gamma_i$. Taking the norm of both members of equation (1), and recalling that $a_i+b_i h_i=0$, we have $\| f(x_i+h_i) \| = \| c(\xi_i) h_i^2 \| \leqq \gamma_i \| h_i \|^2$. From $h_i = -b_i^{-1} a_i$, we have

$$\| h_i \| \leqq (1/\beta_i)\alpha_i, \tag{2}$$

Thus we may write

$$\alpha_{i+1} \leqq \gamma_i(1/\beta_i)^2\alpha_i^2 \text{ or } (\alpha_i\gamma_i/\beta_i^2)\alpha_i.$$

If the sequence of approximations x_i is to yield a root x' as a limit, it is necessary that the sequence of values of $\| f \|$, viz., $\alpha_{i+1} = \| f(x_{i+1}) \| = \| f(x_i+h_i) \|$ approach zero as a limit, and hence necessary although not sufficient that the limit of $(\alpha_i\gamma_i/\beta_i^2)$ shall not exceed unity. A refinement of these considerations suggests the theorems of the following section.

Justification of Newton's Method.—Theorem 1. *Let x_0 be given and let D be a positive number such that $f(x)$ is expansible in the form* (1) *for every x and $x+h$ (if either be denoted by u) in the domain $\| u-x_0 \| < D$. Let $f'(x)$ be non-singular in the domain, and B be a positive number such that $\| f^{-1}(x) \| < 1/B$ in the domain, and C a positive number such that $\| \frac{1}{2} f''(x) \| < C$ in the domain. Then Newton's Method yields a root x' in the domain, whenever $(CD/B) \geqq \varphi[C \| f(x_0) \| /B^2]$ where $\varphi(\lambda) \equiv \lambda+\lambda^2+\lambda^4+\lambda^8+ \ldots +\lambda^{2^i}+ \ldots$.*

An idea of the behavior of $\varphi(\lambda)$ may be had by noting that $\varphi(\lambda)$ has the unit circle in the complex plane for natural boundary, so that $\varphi(\lambda)$ is not defined for real positive λ's except when λ is not greater than one. Obviously $\varphi(\lambda)$ increases with positive λ's. Some of the numerical values of $\varphi(\lambda)$ are given by the following table.

λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$
0.	0.	0.566126	1.000000
0.1	0.110100	0.6	1.106678
0.2	0.241603	0.7	1.491062
0.3	0.398165	0.8	2.046305
0.356497	0.500000	0.9	3.017540
0.4	0.586255	1.	∞
0.5	0.816421	$\varphi[1/(k+1)] < 1/k, k > 0.$	

The proof of the above theorem is immediate. We shall use the previous notation. From (3), $\alpha_{i+1} \leqq (C/B^2)\alpha_i^2$, and from (2), $\| h_i \| \leqq (1/B)\alpha_i$. Hence $\| \Sigma_0^\infty h_i \| \leqq \Sigma_0^\infty \| h_i \| \leqq (1/B)\,[\alpha_0 + (C/B^2)\alpha_0^2 + (C/B^2)^3\,\alpha_0^4 + \ldots + (C/B^2)^{2i-1}\,\alpha^{2^i} + \ldots\,] \leqq (B/C)\,\varphi[(C/B^2)\,\alpha_0]$. Thus if $\| \Sigma_0^\infty h_i \| = \| (x_0 + \Sigma_0^\infty h_i) - x_0 \| < D$, we have by hypothesis $x' = x_0 + \Sigma_0^\infty h_i$ as a vector in the domain, for which also $\lim_{i=\infty} \alpha_i = 0$.

Theorem 2. *If x' is a root of $f(x)$, and $\| f'(x')^{-1} \|$ exists and is equal to $1/\beta'$ and if $\| \frac{1}{2} f''(x) \| \leqq \beta'/D'$ in the region defined by $\| x - x' \| \leqq D'$ and if $f(x)$ is expansible in the form* (1) *in this region, then there is no other nonequivalent root of $f(x)$ in this region.*

Suppose if possible x'' is another nonequivalent root in the region. Then $f(x'') = f(x') + 'f(x')\,(x'' - x') + \frac{1}{2} f''(\xi')\,(x'' - x')^2$ and since x' and x'' are roots of $f(x)$, the equation $f'(x')\,(x'' - x') = f(x'') - f(x') - \frac{1}{2} f''(\xi')\,(x'' - x')^2$ reduces to $\| x'' - x' \| = \| f'(x')^{-1} \| \,.\, \| [-\frac{1}{2} f''(\xi')]\,(x'' - x')^2 \|$ or $\| x'' - x' \| \leqq (1/\beta')\,(\beta'/D')\,\| x'' - x' \|^2$, so that since $\| x'' - x' \| \neq 0$, $\| x'' - x' \| \geqq D'$, contrary to hypothesis.

The following theorem is a corollary of Theorems 1 and 2 where $D' = 2D$. The proof is trivial and will be omitted.

Theorem 3. *Let x_0 be given and $f(x)$ be expansible in the form* (1) *in the domain $\| \tilde{x} - x \| < B/(2C)$, these being defined as in Theorem 1. If $\| f(x_0) \| \leqq B^2/(3C)$, then Newton's Method yields a root x' in the domain and any root in the domain is equivalent to x'.*

Conclusion.—It may be noted that in these theorems no explicit use is made of h_0, x_1 and similar terms, derivable but not initially given. This feature and the statement of the general Theorems 1, and 2, as against Theorem 3, distinguish the form of these results from those of Dean H. B. Fine, already referred to. The steps in the proofs are in essential taken from the article by Dean Fine, where however the proofs hold only for a finite range, and are here extended to a general range, with a consequent notational simplification. The method here used of obtaining a general result by a mere reinterpretation of the case of one variable, offers several features of novelty and is suggested as, perhaps, of even more interest than the results obtained by its particular application to the present problem.

The present theory applies to nonlinear functionelles and integral equations with quadratic terms. A complete expansion in integral power series is not presupposed. In addition to interpretations of $\| z \|$ as $\sqrt{\Sigma\, z_{(n)}^2}$ and its generalizations, $\| z \|$ may be interpreted as $\max_n \mid z_{(n)} \mid$ or as $\Sigma \mid z_{(n)} \mid$ or in various other ways for the range (1, 0), with obvious extensions to the other cases. We may even with Riesz

(Equations Linéaires) introduce an arbitrary parameter p and write $\| z \| \equiv \sqrt[p]{\Sigma \left| z^{p}_{(n)} \right|}$ as an instance.

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THE COBALTAMMINES

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The metallic elements whose properties of aggregation are such that they have low atomic volumes, or high 'cohesion,' form a series of complex ammonia compounds whose stability seems to increase with the cohesion, or decrease with the atomic volume of the metallic aggregate. These complex ammonia compounds, or ammines are of peculiar interest because, like the hydrates, their structure has not yet been explained in a satisfactory way from the standpoint of ordinary valence theories. Werner, however, has developed a special theory of valence which seems to fit very well the chemical behavior of these particular compounds, whether or not it is in agreement with the general behavior of other chemical substances.

Now perhaps the most important point to be established with respect to a series of salts such as these ammines, is the type of each salt with reference to its ionization in solution. When we turn to the work of Werner and of other investigators in this field, it is found that no work has been done which determines directly the type for these salts, although it might be considered that their molecular conductance as determined by Werner, together with their chemical action, makes it seem probable that they belong to the types according to which he has classified them.

The freezing-point method should give the most easy and certain method for the determination of the type of such salts as these, but it is just here that not only the work of other investigators, but even that of Werner[1] himself, does not agree with his theory. Since in this laboratory there is a double combination potentiometer system specially designed for us by Dr. W. P. White, to give with a fifty-couple copper-constantan thermocouple a reading to one twenty-thousandth of a degree, it seemed advisable to make for the first time a series of accurate freezing-point measurements upon the special series of cobaltammines which was most used by Werner in his work. The analyses were made by a Haber-Zeiss water interferometer, loaned to us by the Geophysical